Bifurcations

We have already seen how the loss of stiffness in a linear oscillator leads to instability. In a practical situation the stiffness may not degrade in a linear fashion, and instability may not lead to solutions that lose stability completely. The behavior of the linear oscillator provides an informative local view of behavior, but in a practical situation we might expect nonlinear effects to limit the response in some way. Bifurcation theory can be used to classify instability phenomena based on generic behavior. In other words, as a control parameter is varied, e.g., the axial load on a structure, what happens to a system as a critical condition is reached?

Clearly the passage of an eigenvalue through to the positive real half plane leads to a qualitative change in the phase portrait, i.e., the behavior of trajectories in the local vicinity of an equilibrium point. As a parameter is (slowly) varied, the response of a system changes (often gradually), but it is the qualitative change in the dynamics that is classified as a bifurcation. Although an elementary classification of bifurcations is based on a one-dimensional description, we will focus attention on two-dimensions (which are fundamentally one-dimensional, based on center manifold theory), since we are primarily interested in oscillations which result from application of Newton's second law to structural (mechanical) systems.

The Saddle-Node Bifurcation

The saddle-node bifurcation is the fundamental instability mechanism of a system under the action of a single control parameter:

$$\dot{x} = \mu - x^2. \tag{45}$$

The control parameter μ and coordinate x are linked quadratically. However, in order to maintain a meaningful relationship with vibration we incorporate this relation into the context of a lightly damped oscillator

$$\ddot{x} + 0.1\dot{x} + x^2 - \mu = 0. \tag{46}$$

Equilibrium corresponds to the rest state and thus

$$x_e = \pm\sqrt{\mu}. \tag{47}$$

The stability of these equilibria can be determined in a number of ways, and we start by considering the oscillations resulting from a small perturbation. Let $x = x_e + \delta$, where δ is a small deviation from equilibrium. Placing this in equation (46), we get

$$\ddot{\delta} + 0.1\dot{\delta} + x_e^2 + 2x_e\delta + \delta^2 - \mu = 0. \tag{48}$$

By definition $x_e^2 - \mu = 0$, and neglecting δ^2 (since δ is small) we obtain

$$\ddot{\delta} + 0.1\dot{\delta} + 2x_e\delta = 0. \tag{49}$$

This describes the dynamic response of small perturbations about equilibrium. Substituting in the expression for equilibrium, equation (47), results in

$$\ddot{\delta} + 0.1\dot{\delta} \pm 2\sqrt{\mu}\delta = 0. \tag{50}$$

Taking the positive sign we have a response which oscillates with a frequency a little less than $\omega_n^2 = 2\sqrt{\mu}$. The damping causes the motion to decay back to equilibrium.

Taking the negative sign we have negative stiffness and a solution that grows with time (root structure). The potential energy associated with the saddle-node can be written as

$$V = \frac{x^3}{3} - \mu x, \tag{51}$$

and equilibrium from

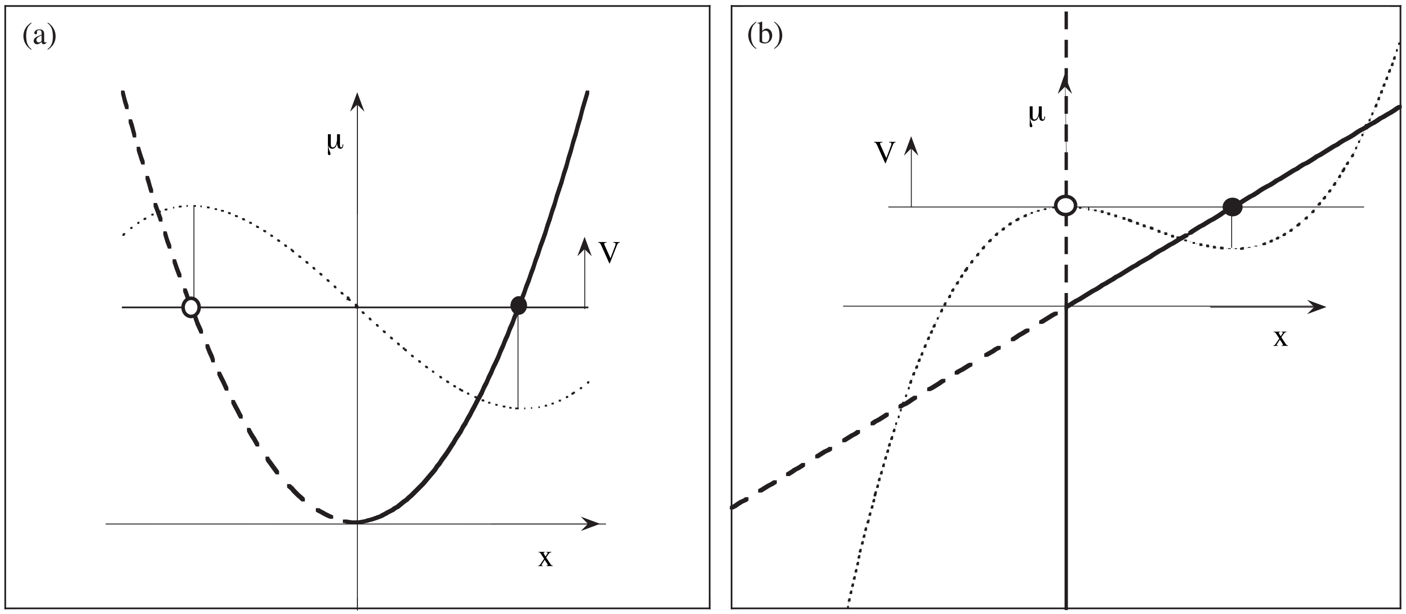
$$V_1 \equiv \frac{dV}{dx} = x^2 - \mu. \tag{52}$$

We have already seen how the sign of the curvature of the potential energy governs stability:

$$V_{11} = 2x, \tag{53}$$

which can be evaluated about equilibrium. When $x_e = \sqrt{\mu}$, the second derivative of the potential energy function is positive indicating that this is a minimum and hence is stable.

A typical graphical representation of this situation is shown below in part (a).



(a) A saddle-node bifurcation, (b) A transcritical bifurcation.

Suppose we have a system with a relatively large positive μ. In this case there is a stable and an unstable equilibrium, characterized by a local minimum and a local maximum of the underlying potential energy respectively. We can imagine the oscillations of a small ball rolling on this potential energy surface (shown dotted in the previous slide). As the value of μ is reduced the two equilibria come together (the frequency of small oscillations will decrease and effective damping increases) as the potential surface flattens out. Just prior to coalescence the stable equilibrium can be thought of as a node, and the unstable equilibrium remains a saddle. Hence their approach (at the critical point) is called a saddle-node bifurcation. No equilibria exist for negative μ and trajectories would simply be swept away. This instability is also sometimes referred to as a fold or limit point.

Bifurcations from a Trivial Equilibrium

Although the saddle-node is the key stability transition in a system under the action of a single control, there are many systems in mechanics in which some kind of initial symmetry is present. An example is the transcritical bifurcation. In the context of a second order ordinary differential equation we can write

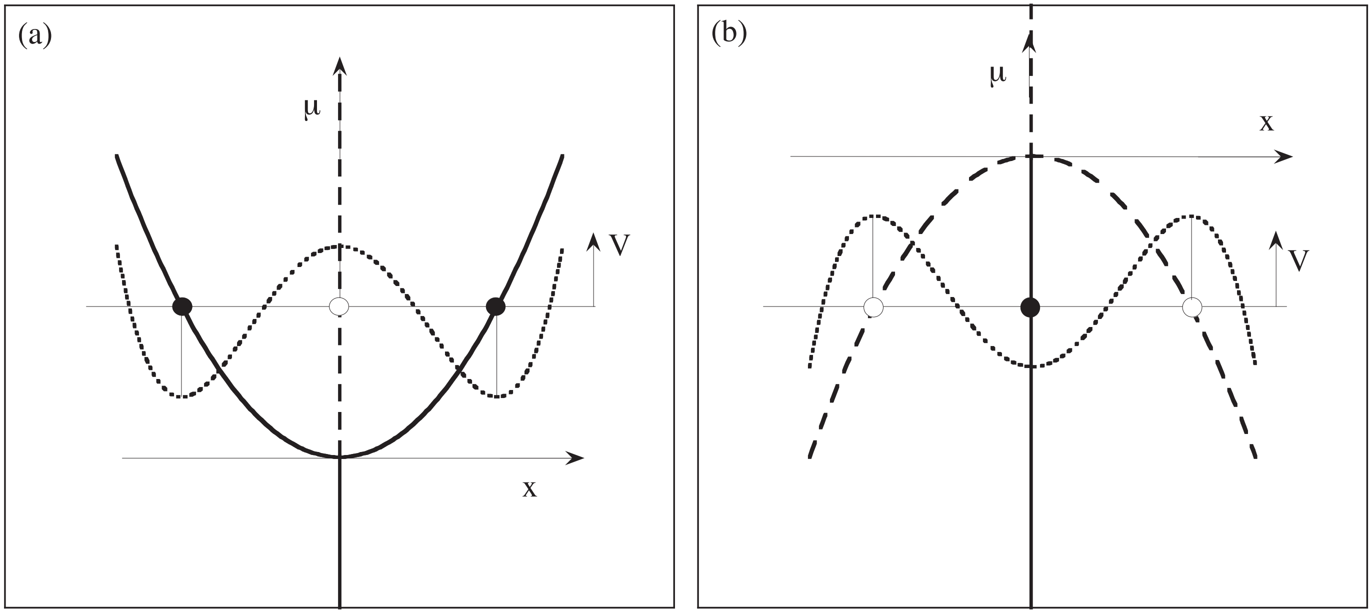
$$\ddot{x} + 0.1\dot{x} + x^2 - \mu x = 0. \tag{54}$$

Following the same approach as for the saddle-node we obtain the situation illustrated in part (b) of the previous figure. Here, there is a fundamental (trivial) equilibrium for negative μ which loses stability as μ passes the through the origin (from negative to positive). The other equilibrium becomes stable at this point and deflection occurs in the positive x direction.

The final pair of bifurcations are associated with the loss of stability of the trivial solution and have global symmetry. They represent an important class of instability in structural mechanics: super- and sub-critical pitchfork bifurcations. For the super-critical pitchfork bifurcation we can consider the oscillator

$$\ddot{x} + 0.1\dot{x} + x^3 - \mu x = 0. \tag{55}$$

Again, we observe the $x_e = 0$ solution, which is stable for $\mu < 0$. At $\mu = 0$ a secondary equilibrium intersects the fundamental and it can be shown that the two (symmetric) non-trivial solutions are stable. This situation corresponds to the classic double-well potential which is also shown superimposed for a specific (positive) value for μ.



(a) A super-critical pitchfork bifurcation, (b) A sub-critical pitchfork bifurcation.

The corresponding sub-critical pitchfork bifurcation:

$$\ddot{x} + 0.1\dot{x} + x^3 + \mu x = 0, \qquad (56)$$

is shown in part(b).

In this case, suppose we start from a negative value of μ. The trivial equilibrium is again stable but now, when the critical point is reached, the system becomes completely unstable. Furthermore, as the critical point is approached, the potential energy maxima associated with the adjacent saddles start to erode the size of allowable perturbations. This is an important consequence of the nonlinearity in the system. Although these last two bifurcations have the same stable trivial equilibrium and critical point, they have quite different consequences if encountered in practice. Hence, they are sometimes characterized as safe or unsafe according to whether a local, or adjacent, post-critical stable equilibrium is available.

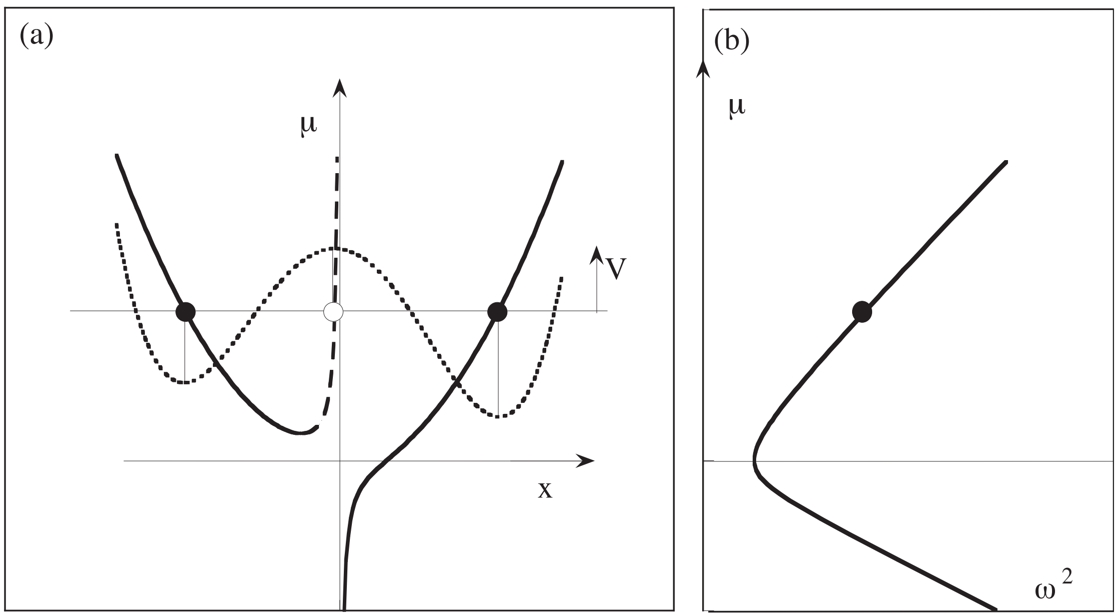
Initial Imperfections

It has already been mentioned that initial geometric imperfections or load eccentricities may have a relatively profound effect on stability. We shall consider this type of effect and its influence on the super-critical pitchfork. Incorporating a small offset causes equation (55) to be altered to

$$\ddot{x} + 0.1\dot{x} + x^3 - \mu x + \epsilon = 0, \tag{57}$$

where ϵ is a small parameter, which breaks the symmetry.

Part (a) shows how the instability transition is changed.



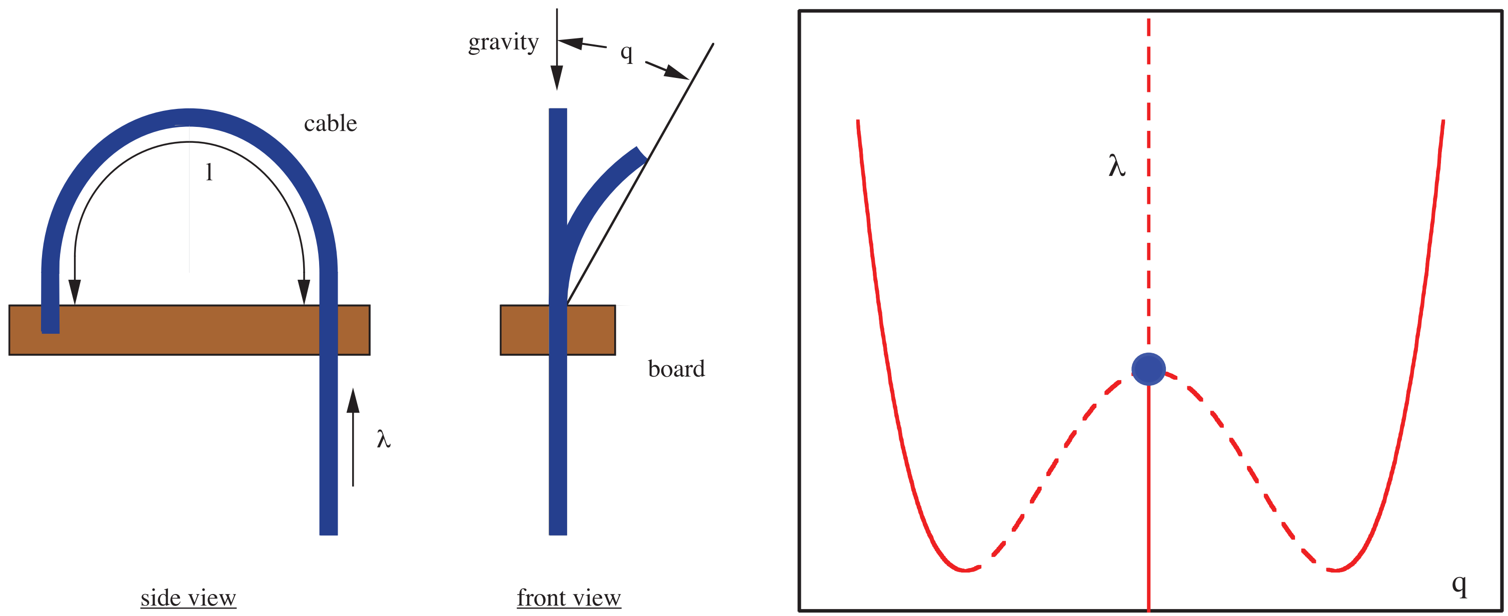
(a) A perturbed super-critical pitchfork bifurcation, (b) Corresponding natural frequency (for the primary branch).

Now, for large negative μ we have a primary equilibrium slightly offset from $x = 0$, and this simply grows as μ approaches, and then passes beyond, the critical value for the perfect geometry (the origin). There is also a complementary solution for negative x, but this wouldn't ordinarily be accessed under a natural loading history, i.e., as μ is monotonically increased. However, the complementary solution does possess a critical point, and this is actually a saddle-node bifurcation (which would be encountered if μ was initially large and x negative, and then μ were reduced). We also note the small tilt in the potential energy function. Furthermore, the complementary solution has an effect on very large amplitude motion and strong disturbances, and this will be revisited later.

Initial imperfections have little effect on the saddle-node bifurcation, but have an especially important influence on the sub-critical pitchfork bifurcation which is termed imperfection sensitive, i.e., the magnitude of the critical load is considerably reduced in the presence of imperfections. For the trans-critical bifurcation the reduction of the maximum critical load occurs for some imperfections but not all.

A Simple Demonstration Model

Consider the slender, flexible system shown below (attributed to Brooke Benjamin).



Schematic of an example bifurcation problem.

This simple system exhibits an unstable-symmetric (sub-critical pitchfork) bifurcation which subsequently stabilizes for large deflections.

We shall conduct a qualitative analysis of this system by associating a load parameter, μ, with the length of the cable (measured from the critical value), and a displacement, q, associated with a general out-of-plane deflection.

The flexible cable is somewhat unusual since it possesses a moment-curvature relation that exhibits a softening spring effect, i.e., the sub-critical bifurcation manifests itself as a sudden motion from in-plane to a drooped out-of-plane position.

A qualitative form of the underlying potential function for this system can be written as

$$V = \frac{1}{720}q^6 - \frac{1}{24}q^4 - \frac{1}{2}\mu q^2. \tag{58}$$

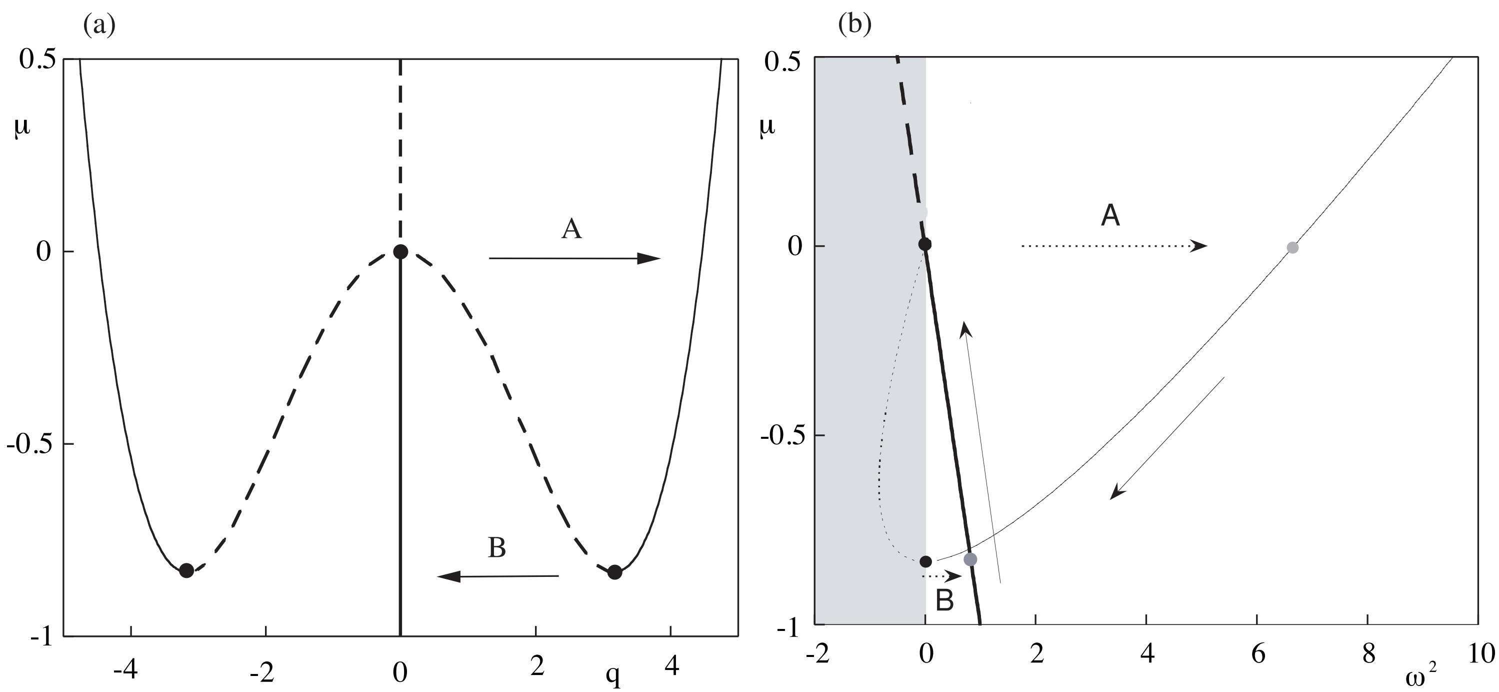
That is, a function reflecting the global symmetry of the system and anticipated equilibria, which are given by the solutions of

$$V_1 = \frac{1}{120}q^5 - \frac{1}{6}q^3 - \mu q = 0. \tag{59}$$

Thus equilibrium curves are given by:

$$\begin{aligned} q &= 0 \\ \mu &= -\frac{1}{6}q^2 + \frac{1}{120}q^4, \end{aligned} \tag{60}$$

and are shown on the next page in part (a).



Equilibrium (a) and dynamics (b) of the wire for the initially perfect geometry.

The second derivative of potential energy is

$$V_{11} = \frac{1}{24}q^4 - \frac{1}{2}q^2 - \mu = 0, \tag{61}$$

and evaluating this expression along the equilibrium paths of equation (60) we get

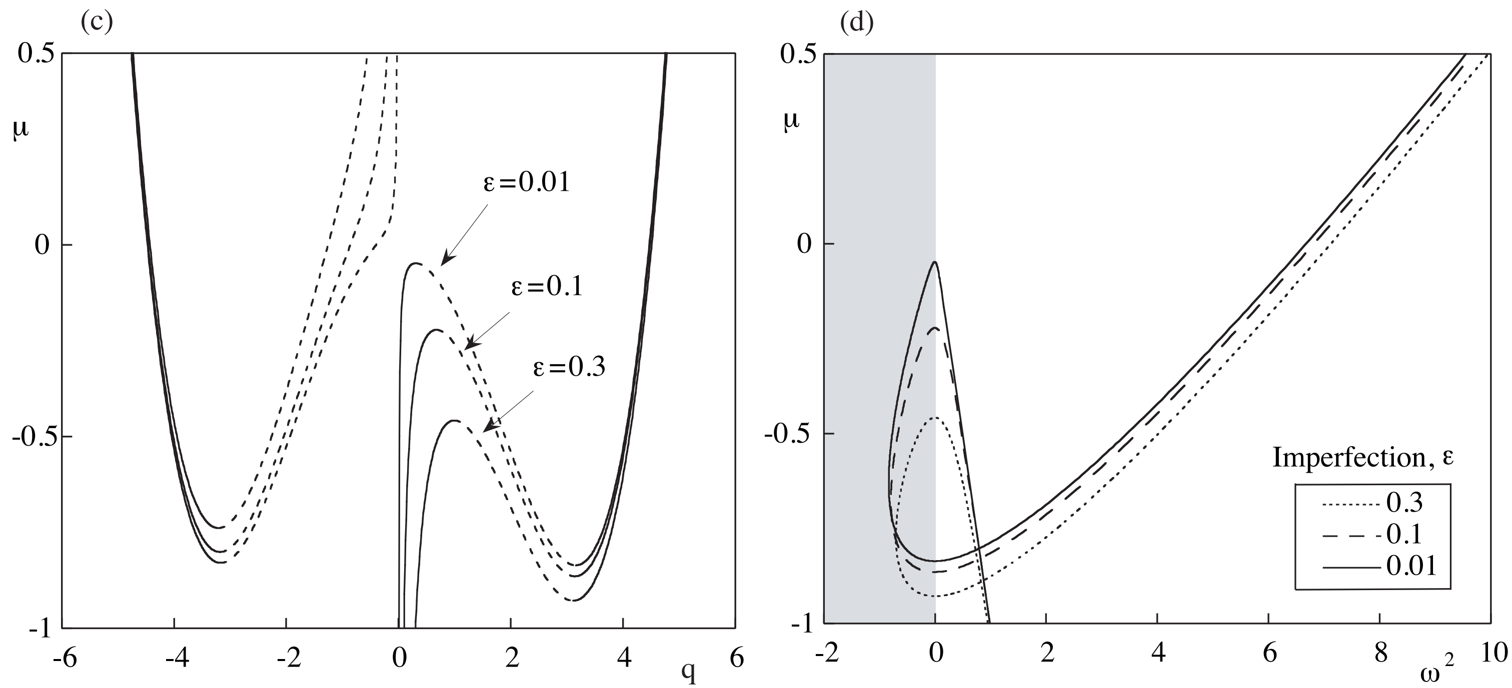
$$\begin{aligned} V_{11}^f &= -\mu \\ V_{11}^p &= \frac{1}{24}\left[10 \pm \sqrt{100 + 120\lambda}\right]^2 \\ &\quad -\frac{1}{2}\left[10 \pm \sqrt{100 + 120\lambda}\right] - \mu. \end{aligned} \tag{62}$$

The sign of these expressions thus indicate stability. Assuming a simple quadratic form for the kinetic energy we can view equations (62) as representing the natural frequencies. These are plotted as a function of the control (the length of the cable) in part (b). We see a linear decay as the critical value is reached, followed by a finite jump to a higher frequency associated with the heavily drooped equilibrium.

An initial imperfection can again be incorporated into the analysis starting from

$$V = \frac{1}{720}q^6 - \frac{1}{24}q^4 - \frac{1}{2}\mu q^2 + \epsilon q. \tag{63}$$

Plots of these relations are shown in the figure below for some representative initial imperfections.



Equilibrium (a) and dynamics (b) of the wire for the initially imperfect geometry.