Part D: Frames and Plates

A Beam with General Boundary Conditions

The general expression governing axially-loaded beam vibrations:

$$EI\frac{d^4W(x)}{dx^4} + P\frac{d^2W(x)}{dx^2} - m\omega^2 W(x) = 0. \tag{1}$$

Again, the mode shapes are given by

$$W(\bar{x}) = c_1 \sinh M\bar{x} + c_2 \cosh M\bar{x} + c_3 \sin N\bar{x} + c_4 \cos N\bar{x}, \tag{2}$$

with $\bar{x} = x/L$, and where M and N are given by

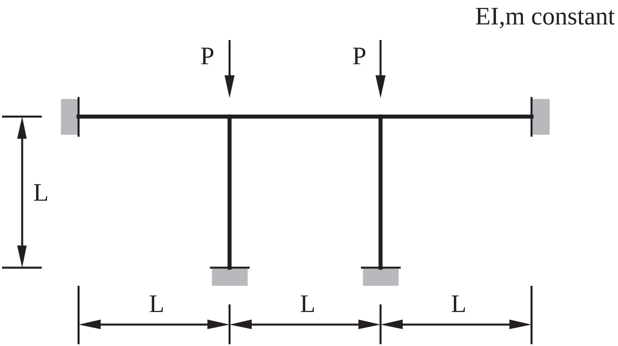
$$\begin{aligned} M &= \sqrt{-\Lambda + \sqrt{\Lambda^2 + \Omega^2}} \\ N &= \sqrt{\Lambda + \sqrt{\Lambda^2 + \Omega^2}}. \end{aligned} \tag{3}$$

The axial load and natural frequency parameters can be conveniently nondimensionalized using:

$$\Lambda = PL^2/2EI, \quad \Omega^2 = m\omega^2 L^4/EI. \tag{4}$$

Now, instead of applying one of the simple boundary conditions, e.g., clamped ($W = dW/dx = 0$), or relating the end displacement to an elastic torsional spring constant ($d^2W/dx^2 - (k_1 L)/(EI)dW/dx = 0$), we have displacement and rotations at the ends that depend on the applied shear forces and moments according to standard beam theory.
Now the coefficients in these relations depend on involved combinations of trigonometric and hyperbolic functions of M and N. Certain conditions of continuity are then used to ensure, for example, that the moments at a common joint balance. Rather than write out the general expressions, we illustrate the process with an example.

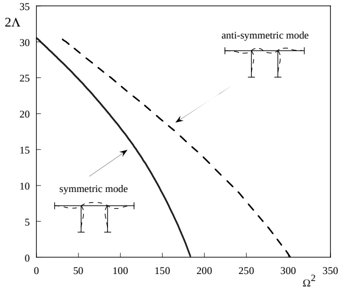
Consider the frame shown below:



A symmetrically-loaded plane frame. Prismatic members with the same flexural rigidities (Nowacki).

This is a plane frame with built-in end supports, and thus has only two nodal DOF: the rotations at the two joints where the loads are applied. We assume EI and m are constant throughout

The lowest natural frequency in both symmetric and asymmetric modes are plotted as a function of the axial loading. For the lowest symmetric mode of vibration without axial loads we obtain a natural frequency of $\Omega \approx 13.5$. As the effective natural frequency tends to zero we obtain a critical load coefficient of $2\Lambda_{cr} \approx 30.6$.



The effect of axial loading on the natural frequencies of a plane frame.

This is roughly intermediate between fully fixed (in which case the coefficient of $2\Lambda_{cr} \rightarrow 4\pi^2$) and pinned ($\approx 2\pi^2$).

The Stiffness Method

Although the preceding section furnished an exact solution (at least within the confines of linear beam theory) it was somewhat cumbersome. In most practical structures we will need a more systematic approach, better suited to computational methods, and this is where the finite element method has proven to be so powerful.
Columns and beams make up the "elements" of the system with joints providing a natural selection for "nodes", although in practice a single structural element tends to be subdivided into many sub-elements. Here we focus on a regular thin beam element in which the bending stiffness is influenced by the presence of an axial load and how this affects the dynamics of a plane frame of which it is a part.

We still suppose that the response of a beam can be expressed as

$$w(x,t) = W(x)Y(t). \tag{5}$$

Then we still have the potential energy of flexural deformations given by

$$V = \frac{1}{2}EI\int_0^L [W''(x)]^2 dx. \tag{6}$$

Furthermore, we also assume that the transverse deflections can be described by a cubic polynomial

$$W(x) = a_1 w_1 + a_2\theta_1 + a_3 w_2 + a_4\theta_2 \tag{7}$$

where

$$\begin{gathered} a_1 = 1 - 3(x/L)^2 + 2(x/L)^3 \\ a_2 = x[1 - 2(x/L) + 2(x/L)^2] \\ a_3 = (x/L)^2[3 - 2(x/L)] \\ a_4 = x^2/L[-1 + (x/L)] \end{gathered} \tag{8}$$

and subscripts 1 and 2 (on the nodal deflections) refer to the left $x = 0$ and right $x = L$ ends of the beam respectively.

That is, we assume the nodal deflections (w, θ) and forces are related by a displacement, or shape, function (that satisfies the boundary conditions). This expression can then be used to obtain the strains, and the principle of virtual work used to evaluate the force corresponding to coordinate i due to a unit displacement of coordinate j, given by

$$k_{ij} = EI \int_0^L W_i''(x) W_j''(x) dx, \tag{9}$$

and hence the element stiffness matrix

$$[K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}. \tag{10}$$

This matrix can also be derived from the traditional slope-deflection equations. It is symmetric and positive definite.

Similarly we recall that the strain energy due to (constant) axial loading on a beam of length L is given by

$$V_p = -\frac{P}{2}\int_0^L [W'(x)]^2 dx, \tag{11}$$

and following the same arguments and using the same shape functions we obtain the consistent geometric influence coefficients

$$k_{Gij} = P\int_0^L W_i'(x) W_j'(x) dx, \tag{12}$$

and hence the geometric stiffness matrix

$$[K_G] = \frac{P}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}. \tag{13}$$

A consistent mass matrix formulation, using the same deflected shape function, equation (7), leads to the mass matrix

$$[M] = \frac{\rho A L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}. \tag{14}$$

Thus, assuming harmonic motion in the usual way, we can arrive at the eigenvalue problem for a prismatic beam:

$$\left[[K] + \lambda [K_G] - \omega_i^2 [M] \right] W_i = 0 \tag{15}$$

in which λ is the load factor, i.e., the factor by which P must be increased in order for buckling to occur.

Some simple beam examples.

Consider a cantilever beam, clamped at one end and free at the other and subject to an axial load. For the two allowable DOF at the free end we can immediately write down the eigenvalue problem

$$\left| \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \frac{P}{30L}\begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} - \frac{\omega^2 \rho A L}{420}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix} \right| \quad (16)$$

The roots of this 2×2 determinant give the natural frequencies at different axial load levels P.

In the usual way, we have natural frequencies in the absence of axial load given by

$$\omega_1 = 3.533 \left(\frac{EI}{\rho A L^4}\right)^{1/2}, \quad \omega_2 = 34.81 \left(\frac{EI}{\rho A L^4}\right)^{1/2}, \tag{17}$$

for which the corresponding exact coefficients are 3.516 and 22.03 respectively. As $\omega_1 \to 0$ we obtain the lowest critical load of $P_{cr} = -2.486 EI/L^2$ (the exact coefficient is $-\pi^2/4$, or 2.4674). The relation between ω_1^2 and P is linear.

Suppose we have a prismatic beam clamped at both ends and again subject to a constant axial force of magnitude P. We can model this system conveniently using two elements (of length $l = L/2$). In this case there is just a single DOF (a lateral deflection at the center). Thus formulating the general eigenvalue problem leads to

$$\left[\frac{24EI}{(l/2)^3} + P\frac{72}{30(l/2)} - \omega^2\frac{312\rho A(l/2)}{420}\right] W_i = 0. \tag{18}$$

We immediately have the expression

$$\omega^2 = \frac{420}{312\rho AL}\left[\frac{192EI}{L^3} + \frac{144P}{30L}\right]. \tag{19}$$

Thus, in the absence of an axial load we obtain the natural frequency coefficient $\omega_1^2 = 22.7354$ and a critical load of $P_{cr} = -40$. Both of these are relatively close to the exact answers (22.3729 and $-4\pi^2$ or -39.4784). Dividing the beam into more elements improves the accuracy and furnishes higher frequencies and critical loads. Other boundary conditions can be handled very simply, with the allowable nodal deflections determining the size of the system of simultaneous equations to be solved. Since the stiffness method is based on Rayleigh-Ritz the results generated are upper bounds on the magnitudes of the frequencies and buckling loads.

Using this approach the finite element packages ANSYS and ABAQUS were able to quickly reproduce the results just described. This approach becomes particularly useful when the geometry of the structure becomes more complex. When individual members connect at various angles then an important step in the analysis concerns transformation between local and global coordinate systems. For example, let's reconsider the plane frame from earlier. Using appropriate nondimensionalized values, a standard ANSYS analysis resulted in the following table:

Load (P)	Ω_1^2	Ω_2^2
0	183.7303	302.3893
6	162.7507	264.0078
12	135.3487	215.7064
18	100.0943	159.9573
24	56.4045	99.1286
30	4.9596	34.7882

An independent ANSYS buckling analysis gave the critical elastic load as $P_{cr} = 30.539 (\equiv 2\Lambda)$ (which compares with the analytical result 30.6). One of the advantages of FEA is the ease with which parameters can be changed and cases rerun. For example, suppose the side support points were pinned rather than fixed. In this case, it is easy to show that the buckling load (in sway) is reduced to $p_{cr} = 8.04$, i.e., we have a more flexible frame with less load carrying capacity and lower natural frequencies.

Thin Plates

The foregoing material can also be developed for 2-dimensional plates, but the analysis quickly becomes cumbersome.
However, we can illustrate a relation between the natural frequencies of vibration and axial load for a flat rectangular panel under bi-axial compression with simply-supported boundary conditions on all four sides.
We assume a deflection for the plate that satisfies the boundary conditions of zero deflection and bending moment along the edges:

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad at \;\; x = 0, a \tag{20}$$

$$w = \frac{\partial^2 w}{\partial y^2} = 0 \quad at \;\; y = 0, b. \tag{21}$$

It is important to note that for this kind of small deflection analysis only the transverse (out-of-plane) boundary conditions are required.

For harmonic motion we have the typical form of response

$$w(x, y, t) = W(x, y) \sin \omega t \tag{22}$$

which is substituted into the governing equation to give

$$D\nabla^4 W - \mu\omega^2 W = N_x \frac{\partial^2 W}{\partial x^2} + N_y \frac{\partial^2 W}{\partial y^2}. \tag{23}$$

We assume a form of spatial response (which satisfies the boundary conditions):

$$W(x, y) = \sum_{m,n=1}^{\infty} A_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{24}$$

Substituting this in equation (23) we get

$$\mu\omega_{mn}^2 = D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 + N_x\left(\frac{m\pi}{a}\right)^2 + N_y\left(\frac{n\pi}{b}\right)^2. \tag{25}$$

Let's initially consider the special case of uniaxial loading, i.e., $N_x = -N, N_y = 0$. In this case we can simplify equation (25) to

$$\begin{aligned} \omega_{mn}^2 &= \frac{D}{\mu}\left[\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)^2 - \frac{N}{D}\left(\frac{m\pi}{a}\right)^2\right] \\ &= \frac{D\pi^4}{\mu a^4}\left(m^2 + n^2 k^2\right)^2 - \frac{N}{\mu}\left(\frac{m\pi}{a}\right)^2, \end{aligned} \tag{26}$$

where $k = a/b$ is the aspect ratio of the plate.

In order to clarify the relation between the natural frequencies and the axial loading we nondimensionalize in the following way. When $N = 0$ we have the natural frequencies given by

$$\bar{\omega}_{mn}^2 = \frac{D\pi^4}{\mu a^4}\left(m^2 + n^2 k^2\right)^2, \tag{27}$$

and when the natural frequencies are zero we have the buckling loads

$$N_{cr} = D\left(\frac{\pi}{ma}\right)^2 (m^2 + n^2 k^2)^2, \tag{28}$$

and thus equation (26) can be written

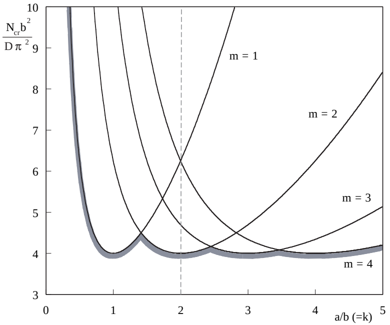
$$\left(\frac{\omega_{mn}}{\bar{\omega}_{mn}}\right)^2 = 1 + \frac{N}{N_{cr}}. \tag{29}$$

Again we obtain a linear relation between the square of the natural frequencies and the axial load for each mode: we expect to see the lowest natural frequency dropping to zero as the lowest buckling load is approached. The linearity of this relation again depends on the similarity between the vibration and buckling modes. The lowest buckling load depends on the aspect ratio in the following way.

Rewriting equation (28) in the form

$$N_{cr} = D\left(\frac{\pi}{b}\right)^2\left(\frac{m}{k} + \frac{n^2 k}{m}\right)^2, \qquad (30)$$

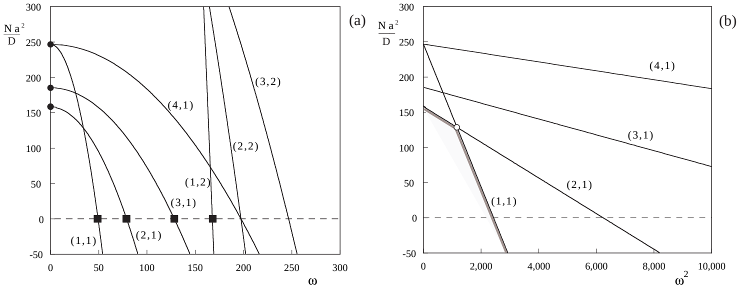
where the aspect ratio is $k = a/b$. We seek the smallest value of this function to give the critical load. Clearly this occurs when $n = 1$, i.e., a half sine wave in the shorter direction, but plotting equation (30) for different integer values of m and hence altering the aspect ratio k leads to the plot shown below:



Buckling loads versus aspect ratio for a simply-supported, uni-axially compressed plate.

Therefore, given a certain aspect ratio, this graph will give the critical load together with the number of half sine waves in the x-direction (which is the direction of the loading in this example). A couple of features of this plot are noteworthy: buckling cannot take place for nondimensional axial loads less than 4 (for any aspect ratio), and the aspect ratio corresponding to the transition in buckling from mode m to mode $m+1$ occurs at an aspect ratio of $\sqrt{m(m+1)}$.

The behavior dependence on the aspect ratio also allows us to re-interpret the simple linear relation from equation (29).



The relation between the natural frequencies and axial load for a uniaxially-loaded, simply supported plate with an aspect ratio of 2.

The sequence of buckling loads, and natural frequencies in the absence of axial loading, are indicated by the solid circular and square symbols in the above figure (a), respectively.

Thus, the fundamental frequency will always correspond to the mode shape with $n = 1$ but not necessarily $m = 1$. For the plate with an aspect ratio of 2 this change-over occurs at 80% of the buckling load (and indicated by the open circle in part (b)). It can also be shown that for a plate with an aspect ratio of 3, the fundamental mode changes from (1,1) to (2,1) at 63% of its critical load and from (2,1) to (3,1) at 85% of its critical load. This is quite different from the beam. We also note that the critical load scales with the cube of the plate thickness, and the natural frequency with the $3/2$ power of the thickness.

Initial Imperfections

As for struts, the behavior of axially-loaded plates may be profoundly affected by the presence of initial geometric imperfections, especially in the vicinity of critical loading conditions. These can be incorporated into the plate theory by appropriately including the total deflection, made up of an initial deflection w_0 together with additional deflection w_1 caused by the application of a transverse load, say. The total deflection is then $(w = w_0 + w_1)$, and this is used in the governing equation or strain energy.

As a simple example, we go back to consider the simply supported rectangular plate under uniaxial compression N. We assume the plate has an initial curvature given by

$$w_0 = Q_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}, \tag{31}$$

in which Q_0 is understood to be small. The plate deflection is $w = w_0 + w_1$ and its maximum value (at the center-point) will then be given by

$$w_{max} = \frac{Q_0}{1-\alpha}, \tag{32}$$

where

$$\alpha = \frac{N}{\left(\frac{\pi^2 D}{a^2}\right)\left[1 + \left(\frac{a^2}{b^2}\right)\right]^2}. \tag{33}$$

For example, for a square plate we have $a = b$ and a critical load given by $N_{cr} = 4\pi^2 D/a^2$ and thus $N \to N_{cr}$, we have $\alpha \to 1$ and $w_{max} \to \infty$ and we see that the effect of the initial imperfection is to magnify subsequent lateral deflection especially in the vicinity of buckling for the initially perfect plate, a situation similar to the strut.

The Ritz and Finite Element Approaches

Adopting a similar approach to the approximate analysis of axially loaded beams we can also attack problems involving the dynamics of plates by assuming the response, in the usual way, is comprised of a finite number of degrees of freedom

$$w = \sum_{i=1}^{N_i} W_i(x, y) q_i(t), \tag{34}$$

where the $q_i(t)$ are the generalized coordinates (assumed to be harmonically varying in time), and $W_i(x, y)$ are the shape functions satisfying the geometric boundary conditions. In this section it will be convenient to use matrix notation: $w = \lfloor W \rceil^T \{q\}$.

We now return to the expression for strain energy for a plate and write down the internal virtual work in terms of bending moments and the plate curvatures:

$$\delta W_{int} = -\int_A [M_x \delta K_x + M_y \delta K_y + 2M_{xy} \delta K_{xy}] \, dA, \qquad (35)$$

in which $K_x = -\partial^2 w / \partial x^2$, etc. The integrand in equation (35) can also be written in matrix notation, i.e., $\lfloor \delta K \rceil^T \{M\}$. Use is now made of the moment-curvature relations: $\{M\} = [D]\{K\}$, where

$$[D] = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}. \qquad (36)$$

The curvatures are then related to the assumed displacements using $\{K\} = [B]\{q\}$, where

$$[B] = \left[\begin{array}{c} -\lfloor \partial^2 W / \partial x^2 \rceil^T \\ -\lfloor \partial^2 W / \partial y^2 \rceil^T \\ -2\lfloor \partial^2 W / \partial x \partial y \rceil^T \end{array} \right]. \quad (37)$$

Therefore the internal work due to bending can be written as

$$\begin{aligned} \delta W_{int} &= -\int_A \lfloor \delta q \rceil^T [B]^T [D][B]\{q\} dA, && (38) \\ &= -\lfloor \delta q \rceil^T [K]\{q\}, && (39) \end{aligned}$$

and this provides the definition of the stiffness matrix: $[K] = \int_A [B]^T [D][B] dA$.

The virtual work due to stretching can also be rewritten as

$$\delta W_{int} = -\int_A \lfloor \delta\theta \rceil^T [N] \{\theta\} dA, \tag{40}$$

where the slope vector $\theta = \{\partial w/\partial x, \partial w/\partial y\}$ can also be related to the assumed displacements using $\{\theta\} = [G]\{q\}$, where

$$[G] = \begin{bmatrix} -\lfloor \partial W/\partial x \rceil^T \\ -\lfloor \partial W/\partial y \rceil^T \end{bmatrix}. \tag{41}$$

Thus, we can write the internal work due to in-plane stretching as

$$\begin{aligned} \delta W_{int} &= -\int_A \lfloor \delta q \rceil^T [G]^T [N][G]\{q\} dA, && (42) \\ &= -\lfloor \delta q \rceil^T [K_G]\{q\}, && (43) \end{aligned}$$

and this provides the definition of the geometric stiffness matrix: $[K_G] = \int_A [G]^T [N][G] dA$.

A consistent mass matrix formulation can be developed using virtual work and utilizing D'Alembert's principle for the inertia loads to give

$$[M] = \int_A \mu [W]^T [W] dA. \tag{44}$$

Alternatively, the mass can be lumped at discrete locations on the plate.

The choice of shape functions, W_i in equation (34), is a crucial step in this whole process. For a standard Ritz analysis we choose functions satisfying the geometric boundary conditions; for finite elements we often choose low-order polynomials that are then assembled for the whole structure. In either case, there is a wide choice, with FE packages offering a variety of element types. In terms of matrix notation we have

$$[M]\{\ddot{q}\} + \{[K] - \lambda[K_G]\}\{q\} = 0, \tag{45}$$

and assuming harmonic motion, i.e., $\{q\} = \{q_0\}e^{i\omega t}$ we arrive at

$$[\{[K] - \lambda[K_G]\} - \omega^2[M]]\{q_0\} = 0. \tag{46}$$

For nontrivial solutions we set the determinant equal to zero (in practice, for large systems, an iterative solution approach is preferred).

An FEA package also shows the reduction of natural frequencies as a function of axial load, and the table below shows a subset of results, for a simply-supported square plate under uni-axial loading.

Load (% of critical)	Ω_1	Ω_1^2
-200	34.150	1166.2
-100	27.811	773.5
-50	24.048	578.3
0	19.596	384.0
40	15.139	229.2
80	8.661	75.0
90	6.046	36.6
95	3.705	13.7
99	1.410	2.0

This analysis used 128 triangular (non-comforming) elements. The coefficient of the lowest natural frequency at zero axial load is 19.596, and the critical load (at zero frequency) is 3.97, both of which compare well with the analytical values of $2\pi^2 = 19.739$ and 4 from equation (26), respectively. Note that Rayleigh-Ritz and Galerkin would give values no lower than the exact.