Vibration of Axially-Loaded Structures

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Workshop Overview

- A: Introductory Concepts
- B: Discrete Link Models
- C: Beam-Columns
- D: Frames and Plates
- E: Nondestructive Testing
- F: Forced Systems

Part A: Fundamentals

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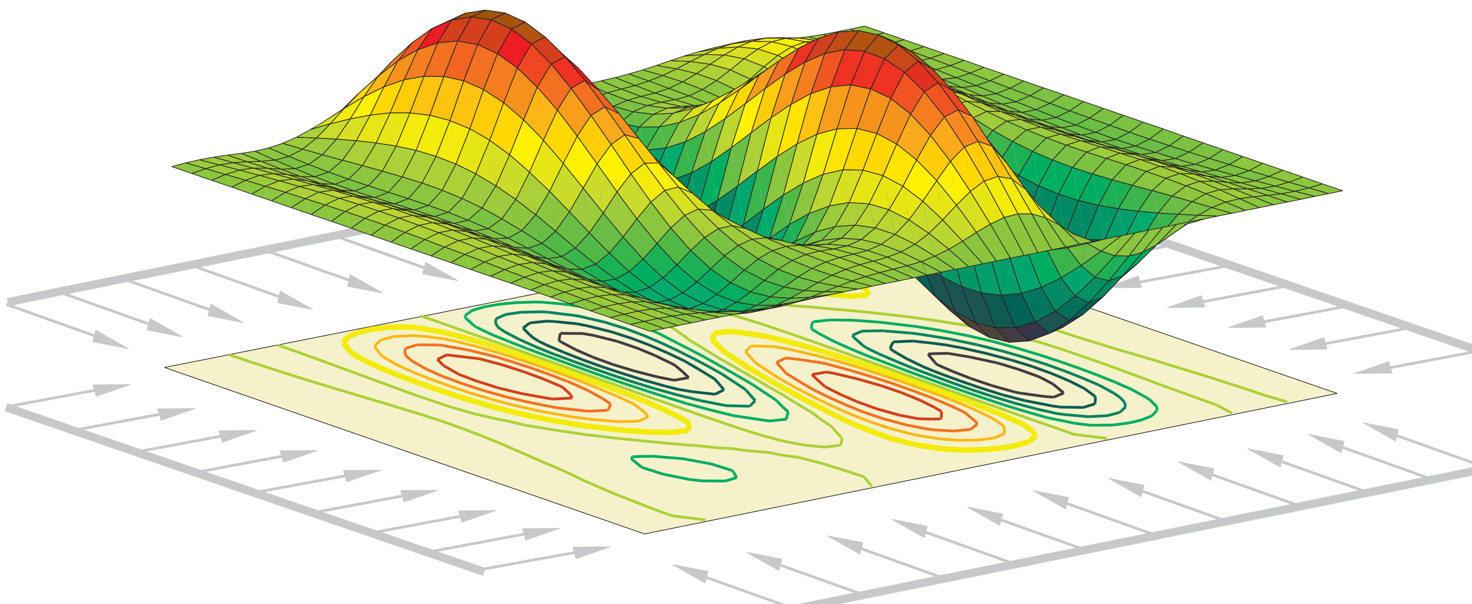
Basic Issues



Preamble

We are all familiar with the classical wave equation for a string:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c_s^2} \frac{\partial^2 w}{\partial t^2}, \qquad (1)$$

where $c_s = \sqrt{\tau/\rho}$ is a constant. Thus the tension τ (a positive force) is an integral part of the system.

We immediately see that negative forces (compression) are not allowed.

However, most structural systems possess bending stiffness. Such systems are often subject to axial loads, and these may be compressive as well as tensile, and in such cases may sometimes exceed the buckling load leading to inherently nonlinear behavior.

Dynamics in the Vicinity of Equilibrium

The Linear Oscillator

Consider the continuous-time evolution of a dynamical system governed by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t) \quad \mathbf{x} \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{2}$$

where $\mathbf{x}$ is a state vector which describes the evolution of the system under the vector field, $\mathbf{F}$. Given an initial condition, typically the values of the state vector prescribed at $t = 0$, i.e., $\mathbf{x}(0)$, we can seek to solve system (2) to obtain a trajectory $\mathbf{x}(t)$, or orbit, along which the solution evolves with time. We will then seek to ascertain the stability of the system, generally as a function of a (control) parameter, μ, and thus consideration of

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mu, t) \qquad \mathbf{x} \in \mathbb{R}^n, \qquad t \in \mathbb{R}, \tag{3}$$

will play a central role in the material to be presented later.

Application of Newton's second law relates acceleration and force (and hence position), and, thus, often results in a second-order ordinary differential equation of the form

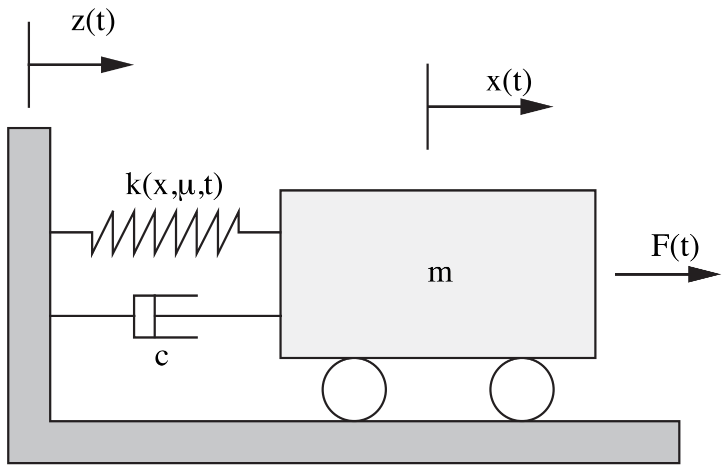
$$\frac{d^2x}{dt^2} = -\omega_n^2 x, \tag{4}$$

where ω_n is a constant (the natural frequency), and with $\dot{x} \equiv dx/dt$, we obtain the nondimensional governing equation of motion

$$\ddot{x} + \omega_n^2 x = 0, \tag{5}$$

subject to the two initial conditions $x(0), \dot{x}(0)$.

This is the equation of motion governing the dynamic response of the spring-mass system shown below with $\omega_n = \sqrt{k/m}$ (k and m constant) and all other parameters set equal to zero, i.e. $c = F(t) = z(t) = 0$.



A spring mass damper.

Since equation (5) is a linear, homogeneous, ordinary differential equation with constant coefficients, we can write the solution as

$$x(t) = Ae^{st}. \tag{6}$$

Placing equation (6) into equation (5) we find that $s = \pm i\omega_n$, and, thus, the general form of the solution is given by

$$x(t) = Ae^{i\omega_n t} + Be^{-i\omega_n t}. \tag{7}$$

Alternatively, using Euler's identities we can write:

$$x(t) = C\cos(\omega_n t) + D\sin(\omega_n t). \tag{8}$$

In order to determine A and B, (or C and D), we make use of the initial conditions to get

$$x(t) = x(0)\cos(\omega_n t) + \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t). \tag{9}$$

This system can be converted into a pair of coupled, first-order ordinary differential equations (in state variable format) by introducing a new variable

$$y = \dot{x} \tag{10}$$

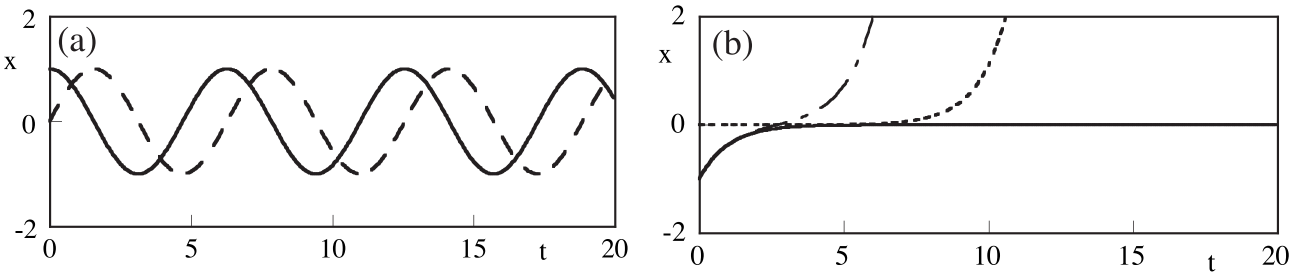
and substituting in equation (5) gives

$$\dot{x} = y, \qquad \dot{y} = -\omega_n^2 x. \tag{11}$$

In matrix notation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{12}$$

A plot of equation (9) (with $\omega_n = 1$) is shown below in (a) for two typical sets of initial conditions.



Time series, (a) Two trajectories exhibiting simple harmonic motion. $x(0) = 1, \dot{x}(0) = 0$ *and* $x(0) = 0, \dot{x}(0) = 1$*, (b) Solutions to equation (17). (i)* $x(0) = -1, \dot{x}(0) = 1$ *(solid line), (ii)* $x(0) = 0.0001, \dot{x}(0) = 0$ *(dotted line), and (iii)* $x(0) = -0.99, \dot{x}(0) = 1$ *(dot-dashed line).*

At this point we simply note that based on equation (9) and its derivative (to obtain velocity) we can envision the trajectory evolving with time in a repeating manner. Plotting position versus velocity (the phase plane) is a useful way of displaying dynamic behavior, and in this (undamped) case it is apparent that the motion is described by ellipses. This is, of course, the periodic behavior we would expect for a simple spring-mass system with ω_n (assumed real, i.e., $\omega_n^2 > 0$) identified as the natural frequency of the oscillation. In terms of a heuristic concept of stability we might consider this behavior to be neither stable or unstable, since any motion we might initiate does not decay or grow, but simply persists.

The solution (8) can also be written as

$$x(t) = \bar{A}\cos(\omega_n t + \phi), \tag{13}$$

where $\bar{A} = \sqrt{C^2 + D^2}$ is the amplitude and $\phi = \arctan\left(C/D\right)$ is the phase. Thus we see that the larger the initial conditions, the larger the area enclosed by the ellipses, i.e.,

$$x^2(t) + (\dot{x}(t)/\omega)^2 = \bar{A}^2. \tag{14}$$

The two trajectories shown earlier differ by a phase $\phi = \pi/2$ and thus the dashed line can be viewed as the corresponding velocity time series. Later, we will see how this relates to energy. However, the form of the resulting motion is independent of the initial conditions.

Suppose we have $\omega_n^2 < 0$. This is a situation that is difficult to envision, physically, but can occur, for example, in a nonlinear system if the spring stiffness becomes negative. Then the motion is governed by

$$\ddot{x} - \omega_n^2 x = 0. \tag{15}$$

Now adopting the solution $x(t) = Ae^{st}$ leads to $s = \pm\omega_n$, and thus

$$x(t) = ae^{\omega_n t} + be^{-\omega_n t}. \tag{16}$$

Using the definition of hyperbolic functions and the initial conditions, we also have

$$x(t) = x(0)\cosh\omega_n t + (\dot{x}(0)/\omega_n)\sinh\omega_n t. \tag{17}$$

In this case we do not have a periodic solution: the positive exponent indicates that typically $x \to \infty$ as $t \to \infty$. Hence, our heuristic concept of stability indicates that this behavior is unstable. However, we also observe that we can choose very specific initial conditions (unlikely but nevertheless important cases), where the trajectory will end up at the origin, i.e., where the positive exponential term is completely suppressed, as well as the case where the negative exponential term in equation (16) dominates for a short time before the trajectory is swept away. These cases were also illustrated earlier.

For all practical purposes, i.e., arbitrary initial conditions, the motion is clearly unstable. The meaning of the special trajectory will be discussed at length later.

Damping

The preceding examples are somewhat unrealistic in terms of practical mechanics since they do not include energy dissipation. With the inevitable presence of damping the question of stability becomes less ambiguous. Typical motion will then consist of a transient followed by some kind of recurrent long-term behavior. This brings us to the fundamentally important concept of an attractor. These are the special solutions alluded to earlier, and they play a key role in organizing dynamic behavior in phase space (the space of the state variables). We shall also see that for nonlinear systems unstable solutions have an important influence on the general nature of solutions.

Suppose we now allow for some energy dissipation in the form of linear viscous damping, i.e., $c \neq 0$ in the SMD. The equation of motion is now

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \tag{18}$$

in which a nondimensional damping ratio, $\zeta \equiv c/(2m\omega_n)$ has been introduced. Solutions to this equation now depend on the value of ζ. For underdamped systems we have $\zeta < 1$ and solutions of the form

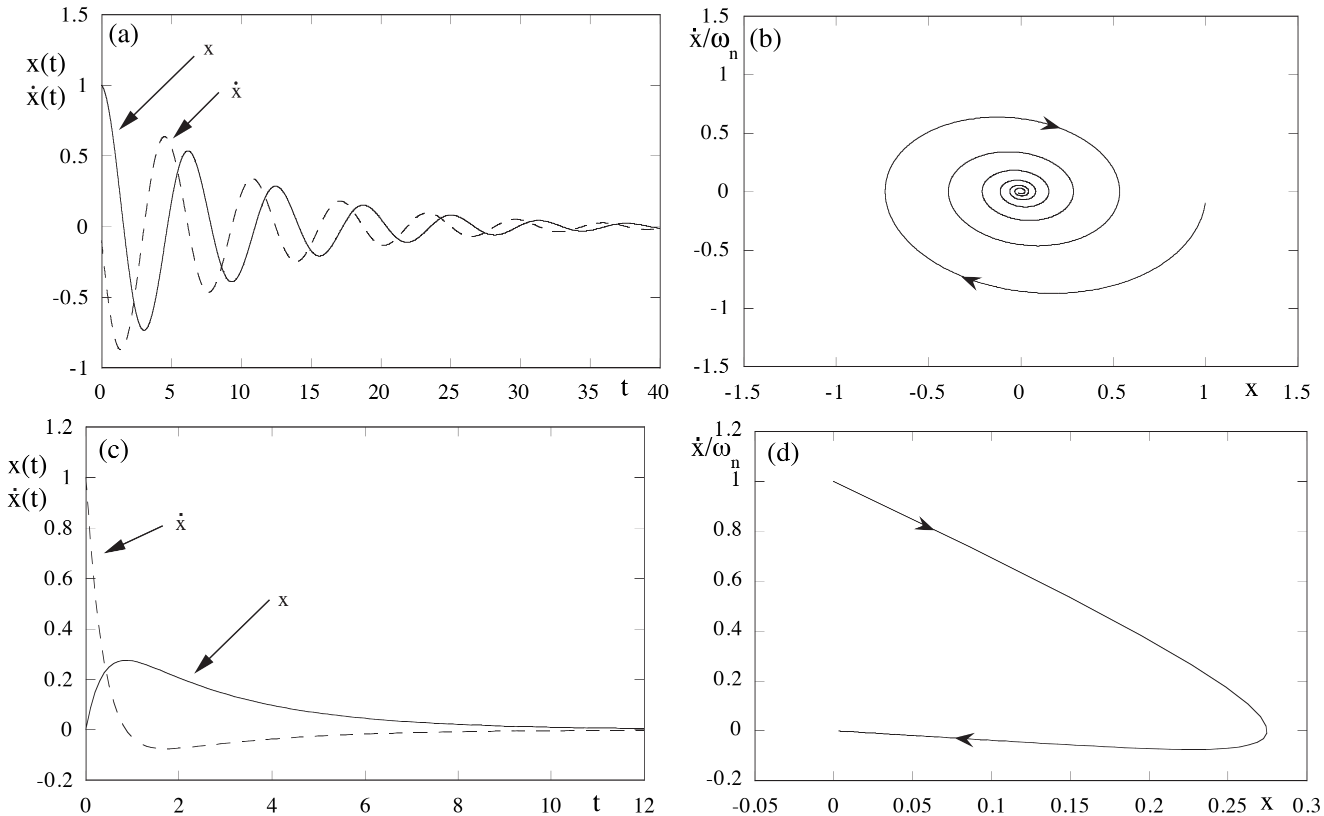
$$x(t) = e^{-\zeta\omega_n t}\left(\frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\sin\omega_d t + x(0)\cos\omega_d t\right), \tag{19}$$

where the damped natural frequency ω_d is given by

$$\omega_d = \omega_n\sqrt{1-\zeta^2}. \tag{20}$$

A typical underdamped response ($\zeta = 0.1$) is shown on the next page as a time series and phase portrait, respectively. The origin indicates a position of asymptotically stable equilibrium,i.e., any disturbance leads to a dynamic response that moves smoothly back to equilibrium. The trajectory gradually spirals down to this rest state: we can imagine a family of trajectories forming a flow as time evolves.

Since this equilibrium is unique, the whole of the phase space is the attracting set for all initial conditions and disturbances. Damping in this range, e.g., ζ 0.1 is quite typical for mechanical and structural systems.



Time series (a) and phase portraits (b) for underdamped (oscillatory) motion, $x(0) = 1.0; \dot{x}(0) = 0.0; \zeta = 0.1$.. (c) and (d) overdamped (non-oscillatory) motion, $x(0) = 0.0; \dot{x}(0) = 1.0; \zeta = 1.5$.

For a heavily (or overdamped) system $\zeta > 1$, and in this case the form of the solution is

$$x(t) = Ae^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + Be^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} \tag{21}$$

where

$$A = \frac{\dot{x}(0) + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x(0)}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{22}$$

and

$$B = \frac{-\dot{x}(0) - (\zeta - \sqrt{\zeta^2 - 1})\omega_n x(0)}{2\omega_n\sqrt{\zeta^2 - 1}}. \tag{23}$$

The motion is a generally monotonically decreasing function of time and may take a relatively long time to overcome rather heavy damping forces on the way to equilibrium. A typical case was also shown earlier.

The boundary between these two cases is the critically damped case, i.e., $\zeta = 1$. We will regularly encounter the situation in which the stiffness of a system degrades, and given the definition of ζ we expect not only a reduction in the natural frequency but also an increase in the damping ratio.

Returning to the state variable matrix format of the linear oscillator we therefore have

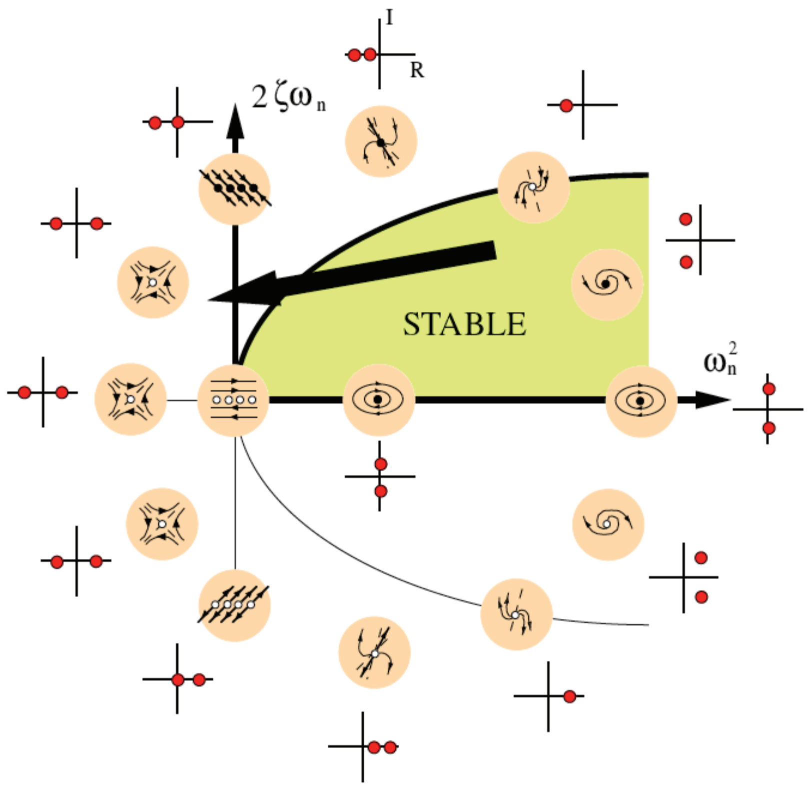
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{24}$$

We can also write the solution in terms of the eigenvalues of the state matrix, i.e., the roots of the characteristic equation

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0. \tag{25}$$

Critical damping thus relates to the discriminant being equal to zero.

Given the scenario of a system losing stability we can usefully view all the response possibilities of this type of linear system according to the location of the roots in the complex plane. For example, having two complex roots with negative real parts corresponds to an exponentially decaying oscillation:



Phase portraits and root structure of a linear oscillator.

In general we will have a system with positive stiffness and damping and thus a root structure corresponding to the upper right quadrant. Critical damping corresponds to the parabola, and phase portraits and eigenvalues are indicated for various combinations of the natural frequency and damping. The system eigenvectors organize the transient behavior in the phase portrait. Some useful terminology here includes the spiral or focus for decaying oscillatory motion (also called a sink), the node for overdamped motion, the inflected node for equal eigenvalues (and thus including the critically damped case), and the saddle for the motion characterized by having both a stable and unstable direction (eigenvector) with instability becoming dominant. We can also view the undamped case as a center.

This is more challenging in the context of higher-order dynamical systems, but more formally we state that if we have a dynamical system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is constant with eigenvalues $\lambda_i, i = 1, 2, ..., n$, then

- (i) If the system is stable, then $Re\{\lambda_i\} \leq 0, i = 1, 2..., n$.
- (ii) If either $Re\{\lambda_i\} < 0, i = 1, 2..., n$; or if $Re\{\lambda_i\} \leq 0, i = 1, 2..., n$ and there is no repeated eigenvalue; then the system is uniformly stable.
- (iii) The system is asymptotically stable if and only if $Re\{\lambda_i\} < 0, i = 1, 2..., n$ (and then it is also uniformly stable, by (ii)).
- If $Re\{\lambda_i\} > 0$ for any i, the solution is unstable.

We thus observe what will typically happen when the stiffness of the system degrades (e.g., due to an axial load acting on a slender structure). For a small amount of damping the eigenvalues start off as a complex conjugate pair with negative real parts. As the stiffness (and hence natural frequency) reduces, the eigenvalues merge on the negative real axis, and then their magnitudes diverge with one entering the positive half-plane. Thus instability occurs, and solutions grow without bound. Although the above description relates to a single-degree-of-freedom (SDOF) linear oscillator this type of scenario is encountered to a large extent within a variety of high order and nonlinear systems. The geometric view afforded by a consideration of the root structure and phase portraits of families of solutions about equilibrium points is very useful. We will make extensive use of linearization to utilize this view locally to equilibrium within a nonlinear context.

An Oscillator with a Slow Sweep of Frequency

Consider again the spring-mass system. Again assume that there is no damping or external forcing ($c = F(t) = z(t) = 0$), and that the spring stiffness decays linearly (in time) from a base value $k = 1$ at $t = 0$. We assume this decay is very slow, and characterized by a small parameter ϵ. In this case we can write the governing equation of motion as

$$\ddot{x} + \mu^2(t)x = 0 \tag{26}$$

in which

$$\mu^2(t) = \frac{k}{m}(1 - \epsilon t), \tag{27}$$

i.e., the system will lose stability when $t \to 1/\epsilon$. If we assume that the evolution of the stiffness change is very slow ($\epsilon << 1$), i.e., much slower than the natural oscillatory response of the system, then we have a number of approaches available for obtaining a solution $x(t)$. A direct numerical solution of equation (26) is easily obtained.

Alternatively, a perturbation approach can be applied, e.g., using the multiple scales technique, we obtain (to the leading term in the expansion)

$$x(t, \epsilon) = x^{+}(t^{+}, \epsilon) = \sqrt{\frac{\mu(0)}{\mu(\tilde{t})}} \left[\cos t^{+}\right], \tag{28}$$

where the initial conditions are taken as unity, the plus sign indicates a solution forward in time, and where

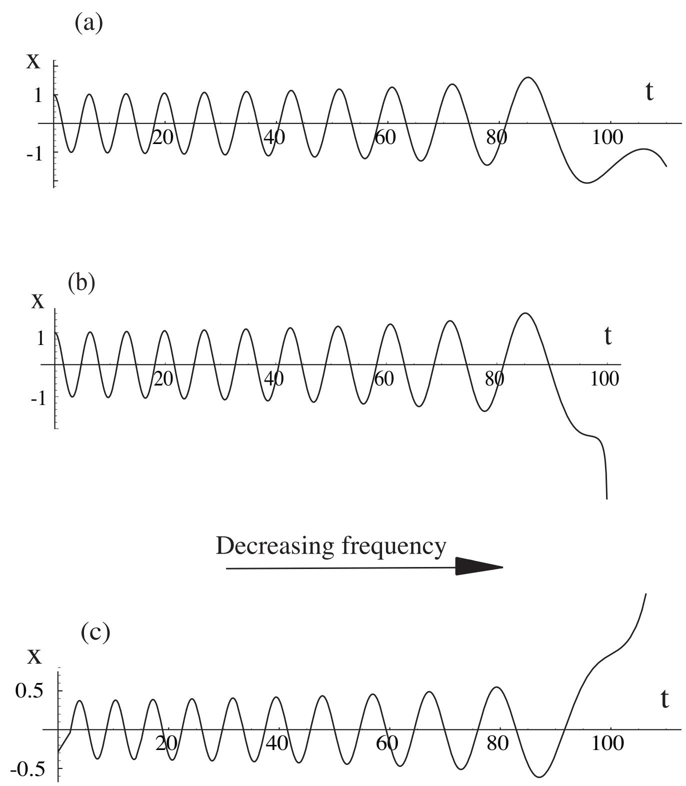
$$t^{+} = \frac{1}{\epsilon} \int_{0}^{\tilde{t}} \mu(\tilde{s}) d\tilde{s}. \tag{29}$$

Inserting the specific form for the linear sweep (equation 27), we get a solution

$$x(t) = (1 - \epsilon t)^{-1/4} \cos\left[-\frac{2}{3\epsilon}(1 - \epsilon t)^{3/2} + \frac{2}{3\epsilon}\right], \tag{30}$$

where $k/m = 1$ has been used.

Consider the specific case of $\epsilon = 0.01$. In this case we would expect the system to lose stability near $t = 100$. Part (c) also illustrates that the frequency and stability characteristics for this type of linear system are not influenced by initial conditions (which are different in this final case).



Some sweeps towards instability. (a) Numerical solution to equation 5, (b) a perturbation solution, (c) an exact solution using Airy functions.

Dynamics and Stability

It is a simple matter to write down the potentialand kinetic energy for the spring mass system

$$V = \frac{1}{2}kx^2, \qquad T = \frac{1}{2}m\dot{x}^2. \tag{31}$$

Applying Lagrange's equation yields the equation of motion. In which we can use $\ddot{x} = \dot{x}d\dot{x}/dx$, separate variables and confirm that energy is conserved. In this simple case both the potential and kinetic energies were positive and quadratic.
But we are especially interested in systems characterized by a potential energy function that is not necessarily quadratic and whose form changes with a (load) parameter.

We also note the relation between energy and the natural frequency

$$\omega_n^2 = \frac{d^2 V}{dx^2} \Big/ \frac{d^2 T}{d\dot{x}^2} \equiv \frac{V_{11}}{T_{11}}, \tag{32}$$

in which the subscripts refer to differentiation with respect to generalized position (for the potential energy) and generalized velocity (for the kinetic energy). This will form the basis of a number of approximate techniques, including Rayleigh's method, to be described later.

Stability Concepts

We have seen how, for conservative, nongyroscopic forces, we can write the potential energy as

$$V = V(Q_i), \tag{33}$$

and Lagrange's equation tells us that the condition for equilibrium (i.e., a stationary state, or no motion) is given by

$$V_i \equiv \frac{\partial V}{\partial q_i} = 0 \tag{34}$$

(for all i), which can be stated in words as

Since we are primarily interested in systems that have a smooth potential energy function, we can develop a Taylor series expansion about equilibrium

$$V = V^E + \sum_{i=1}^{i=n} \frac{\partial V}{\partial Q_i}\bigg|^E q_i + \frac{1}{2}\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} \frac{\partial^2 V}{\partial Q_i \partial Q_j}\bigg|^E q_i q_j + \ldots\ldots, \tag{35}$$

where incremental coordinates $q_i \equiv Q_i - Q_i^E$ have been introduced. Now, if we make use of the tensor summation convention and define

$$\frac{\partial^2 V}{\partial Q_i \partial Q_j}\bigg|^E \equiv V_{ij}^E \tag{36}$$

we obtain the dominant quadratic form

$$V = \frac{1}{2} V_{ij}^E q_i q_j + \ldots\ldots, \tag{37}$$

since $V^E \equiv V(Q_i^E)$ is an arbitrary constant (which we generally choose equal to zero), and the linear term automatically drops out by virtue of equation (34).

So far, we still haven't fully considered stability in terms of energy. Although the notion of equilibrium enables considerable progress to be made in linear structural anlysis, the presence of compressive axial loads demands further scrutiny. A theorem, which goes back to Lagrange, states

For the types of axially-loaded structures of interest here we can write the dominant (quadratic) form of the potential energy as

$$V = \frac{1}{2} V^E_{ij} q_i q_j + ..., \tag{38}$$

$$= \frac{1}{2} \left[U_{ij} q_i q_j - P^k \epsilon^k_{ij} q_i q_j \right], \tag{39}$$

where U_{ij} is the strain energy and P^k is a set of loads with corresponding movement ϵ^k_{ij}. In matrix notation we can also write

$$V = \frac{1}{2} V^E_{ij} q_i q_j = \frac{1}{2} \mathbf{q}^T K \mathbf{q}, \tag{40}$$

in which K is the effective stiffness matrix. For conservative systems this matrix is symmetric.

Similarly, most systems of interest will have a quadratic kinetic energy of the form

$$T = \frac{1}{2} T^E_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{\mathbf{q}}^T M \dot{\mathbf{q}}, \tag{41}$$

in which M is the mass matrix. Placing the general expressions for the energy in Lagrange's equation then yields

$$T^E_{ij} \ddot{q}_i + V^E_{ij} q_i = 0, \tag{42}$$

in terms of the generalized coordinates q_i.

The coordinates can be transformed into principal coordinates u_i such that the equations of motion become decoupled:

$$T_{ii}^E \ddot{u}_i + V_{ii}^E u_i = 0. \tag{43}$$

In this case, we can write down all the natural frequencies:

$$\omega_i^2 = \frac{V_{ii}^E}{T_{ii}^E}. \tag{44}$$

Of course, there are a number of important issues underlying these last few expressions, but considerable research has focused on these types of transformations.