A note on damping

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Incorporating a little viscous damping in our model gives

$$\ddot{q} + \beta\dot{q} + \frac{1}{120}q^5 - \frac{1}{6}q^3 - \mu q = 0. \tag{64}$$

Considering dynamics along the fundamental equilibrium path, i.e., about $q_e = 0$, we have

$$\ddot{q} + \beta\dot{q} - \mu q = 0. \tag{65}$$

Assuming the solution is of the form $q = Ae^{\lambda t}$ we obtain the characteristic equation

$$\lambda^2 + \beta\lambda - \mu = 0, \tag{66}$$

with roots

$$\lambda = -\frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta^2 + 4\mu}. \tag{67}$$

We recall the familiar expression for damped natural frequencies

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}, \tag{68}$$

where $\zeta \equiv \beta/2\sqrt{-\mu}$, i.e., because of the decaying stiffness we observe that the response becomes critically damped just prior to buckling when $\beta = 2\sqrt{-\mu}$. For example, with $\beta = 0.1$ oscillations would cease when $\mu = -0.0025$.

Softening column:
This example can be solved more accurately using classical methods, e.g., as an elastic or using Rayleigh-Ritz: The nondimensional differential equation governing equilibrium is

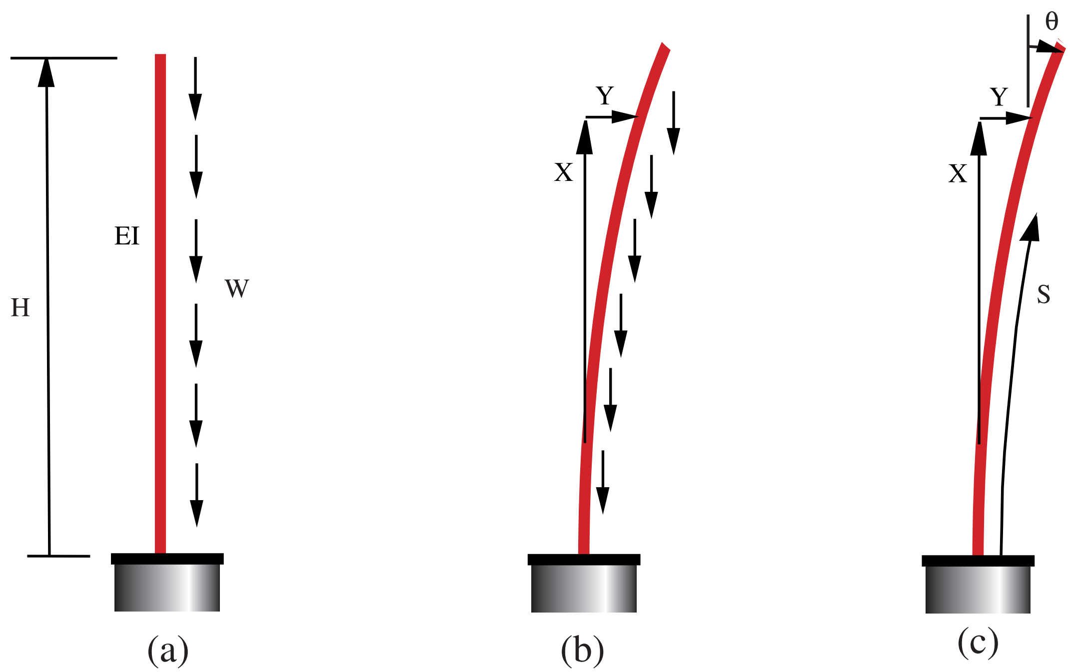
$$y''''(x) + [(h - x)y'(x)]' = 0, \tag{69}$$

with boundary conditions $y(0) = y'(0) = y''(h) = y'''(h) = 0$.
For example, aapproximate expression for the critical height can be obtained using the method of Rayleigh-Ritz and the potential energy function

$$U = \frac{1}{2}\int_0^h (y'')^2 dx - \frac{1}{2}\int_0^h (h - x)(y')^2 dx. \tag{70}$$

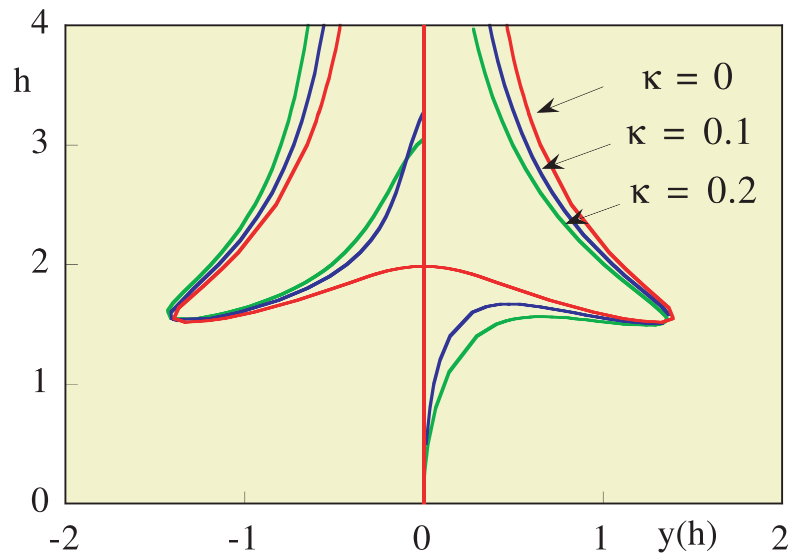
to give solutions very close to the exact value of $h_{cr} = 1.986$.

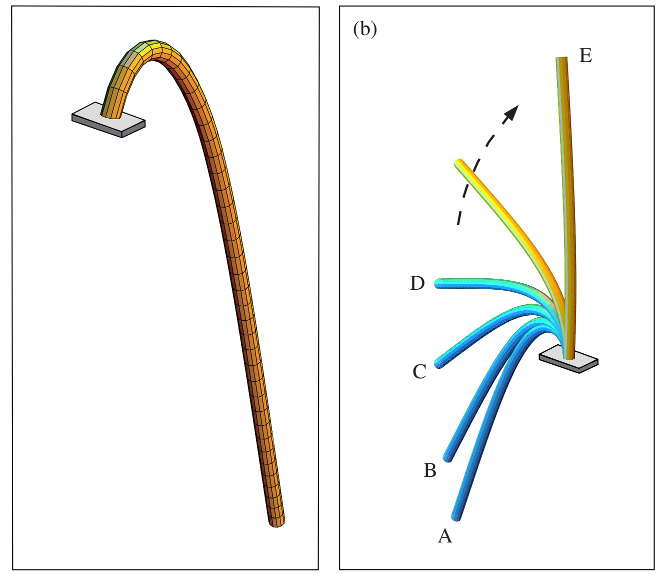
An elastica analysis (incorporating the softening material) can be undertaken:



Equilibrium paths and configurations.

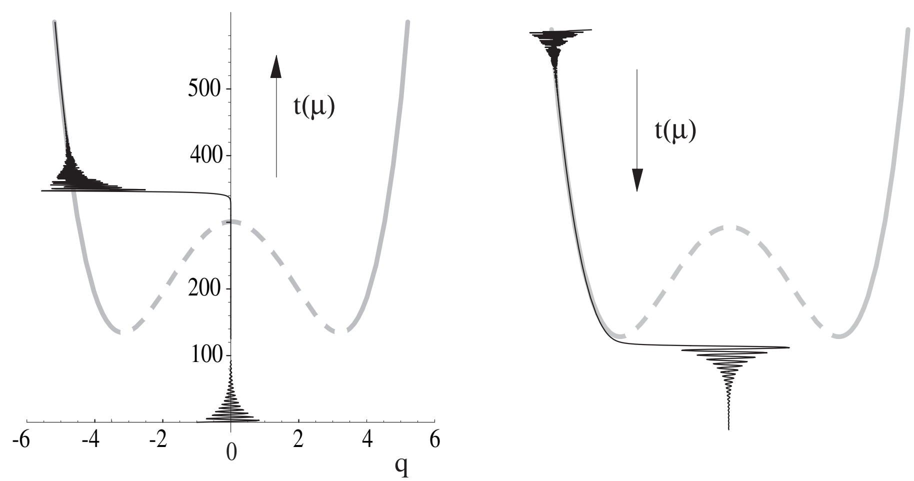
Interest in this problem actually dates back to 1881 (Greenhill) and a classic study into the height to which a tree might grow.



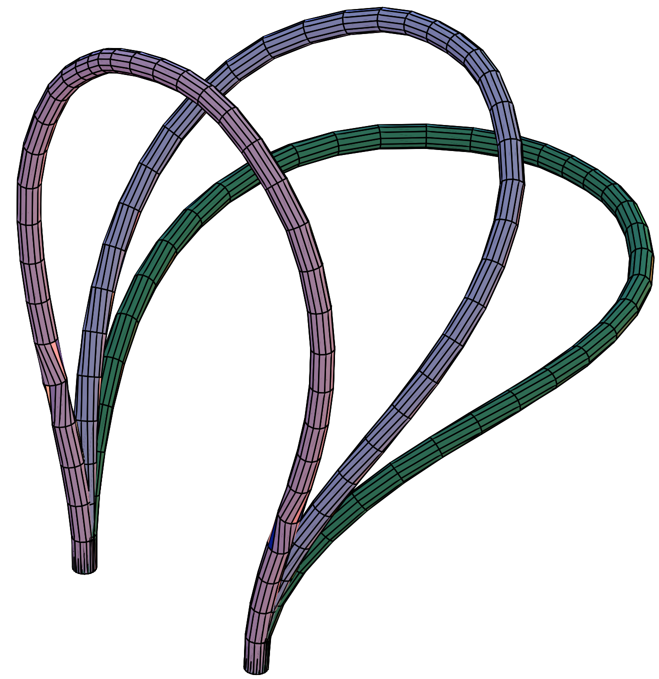


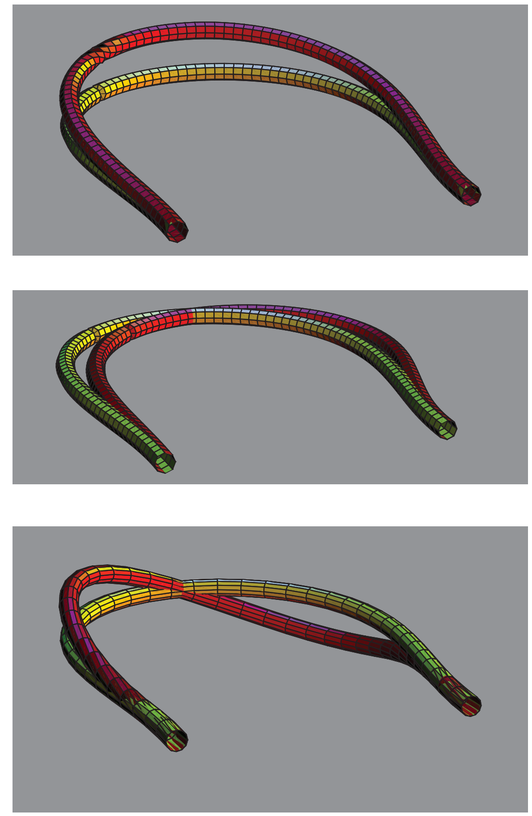
Equilibrium paths and configurations.

The length of the cable can be made to evolve as a linear function of time, and some numerical simulations are shown below, in which the rate of change of the length has been set such that the critical condition is reached after 300 time units. A small overshoot is encountered before the cable rapidly moves to one of its remote equilibria. Also shown in this figure is a reverse sweep when the length of the cable is gradually shortened. A region of hysteresis is readily observed.



A slow sweep through the bifurcations.

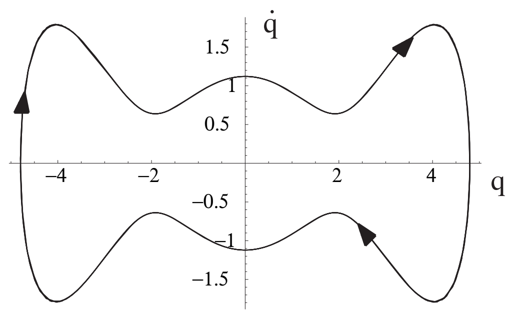




Post-buckled equilibrium configurations and vibrations modes.

'Large' Oscillations

So far, we have been mainly interested in the dynamic behavior in the vicinity of equilibrium. The figure below shows a phase trajectory based on a numerical simulation of the equation of motion when the length of the cable is such that the cable is stable in both the upright position or one of two highly drooped configurations. In this case the damping has been removed and initial conditions set such that the motion traverses across all the equilibrium positions.

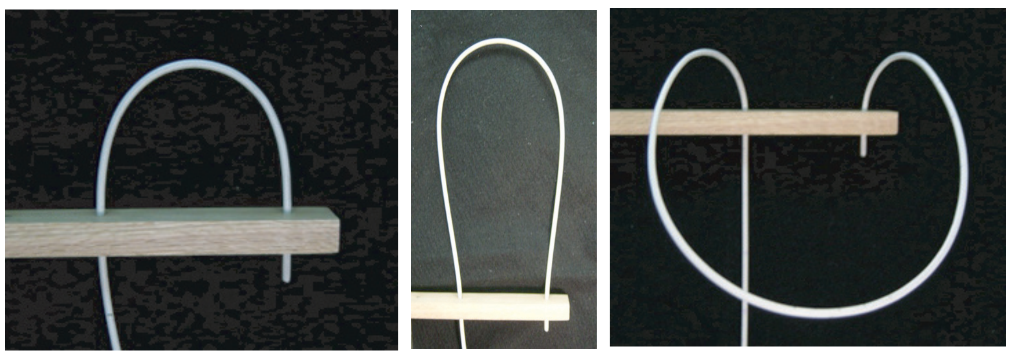


An undamped phase trajectory, $\mu = -0.5, q(0) = 4.8, \dot{q}(0) = 0.$

There are five equilibria at this length: stable at -4.04, 4.04, 0.0 and unstable at -1.92, 1.92.

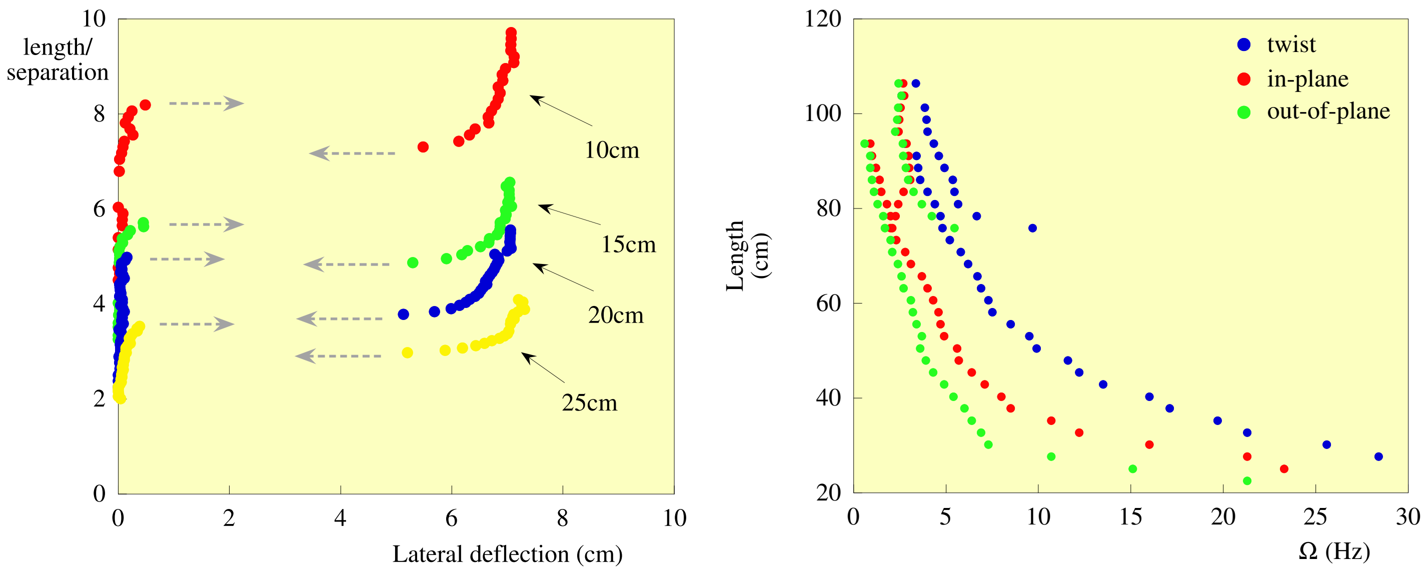
Experiments

This behavior can be shown quite easily in an experiment. Four sets of hole separation were used: 10cm, 15cm, 20cm, and 25cm. For each case the length of the cable was increased and the lateral deflection was measured a short distance up from the base. The experimental system is shown in three stages of deformation below. In a uniform gravitational field the loop will reach a critical value, at which point it flops suddenly to one side, characteristic of a sub-critical pitchfork bifurcation.



Front views of the loop. (a) short arc length, i.e.,$\mu << 0$, prebuckled; (b) longer arc length, i.e., $\mu < 0$, prebuckled; (c) long length, i.e., $\mu > 0$ (or displaced from the trivial equilibrium if in region of hysteresis), postbuckled.

The figure below shows the measured (equilibrium) results. In each case a sudden jump to a severely-drooped configuration is apparent, as well as the hysteresis upon reduction of the cable length. In this case the cable length is scaled by the hole separation. Frequencies were also measured (using a laser velocity vibrometer) and these are also shown (part (b)).



Behavior of loop made of a cable with a softening characteristic. (a) equilibrium paths for four different loops; (b) lowest frequency in each mode, before and after the jump (with an expanded y-axis).